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Translated by A. Y.

STABILITY OF PERIODIC SOLUTIONS OF QUASILINEAR ELASTIC GYROSCOPIC SYSTEMS WITH DISTRIBUTED AND CONCENTRATED PARAMETERS

PMM Vol. 34, №1, 1970, pp. 115-126

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The stability of periodic solutions of quasilinear elastic slightly asymmetric gyroscopic systems with distributed and concentrated parameters is considered. The motion investigated is described by a system of partial differential equations; the boundary conditions and matching conditions at the sites of the concentrated parameters also take the form of quasilinear equations. The nonlinear functions in the equations of motion and in the boundary conditions are assumed to be of sufficiently general form; this makes it possible to investigate the stability of the periodic solutions under the most varied perturbations. It is assumed that some of the natural frequencies of the linearized system can be critical or resonance frequencies. The gyroscopic effect of the distributed mass is assumed to be negligibly small, as usual.

The periodic oscillation states of unbalanced flexible rotors, some of whose supports have nonlinear characteristics, are constructed as an example. The equations in variations are written out and it is shown that their stability can be investigated completely by the proposed method.

1. Many problems of applied mechanics involve the action on quasilinear elastic gyroscopic systems of periodic forces whose frequencies are usually multiples of the angular velocity ω of the gyro system rotor. Their equations of motion have periodic solutions with the period $T = 2\pi / \omega$; however, these solutions may turn out to be unstable for various reasons, so that almost-periodic autooscillatory states (not always permissible ones) arise in the gyro system. The conditions of stability of the periodic oscillations therefore assume considerable importance.

The motion of elastic gyro systems is described in the more complicted cases by a system of nonhomogeneous quasilinear partial differential equations with nonhomogeneous quasilinear boundary conditions. Without limiting generality, let us consider the simplest such system.

Let the functions $u_j(x, t)$, $j = 1, 2$ over the interval $0 < x < l$ satisfy the quasilinear differential equations

$$\frac{\partial^4 u_j}{\partial x^4} + \rho \frac{\partial^2 u_j}{\partial t^2} + \mu\varkappa \frac{\partial u_j}{\partial t} = f_j(x, t) + \mu F_j(\mu, t, x, u_1, u_2, u_{1x}, u_{2x}, u_{1t}, u_{2t}) \qquad (1.1)$$

$$u_j = \partial u_j / \partial x = 0 \quad \text{for } x = 0$$

and

$$\sigma \frac{\partial^2 u_j}{\partial x^2} = -K_1 \frac{\partial^3 u_j}{\partial x \partial t^2} + i^{2j} K_0 \omega \frac{\partial^2 u_{j\pm 1}}{\partial x \partial t} -$$

$$- c \frac{\partial u_j}{\partial x} + g_j(t) - \mu\varkappa_1 \frac{\partial^2 u_j}{\partial x \partial t} + \mu G_j(\mu, t, u_1, \ldots u_{2t}, l) \qquad (1.2)$$

$$\sigma \frac{\partial^3 u_j}{\partial x^3} = m \frac{\partial^2 u_j}{\partial t^2} + g_j^*(t) + \mu\varkappa_2 \frac{\partial u_j}{\partial t} + \mu G_j^*(\mu, t, u_1, \ldots, u_{2t}, l)$$

$$(i = \sqrt{-1},\ j = 1, 2)$$

for $x = l$.

The specified functions f_j, g_j, g_j^*, F_j, G_j and G_j^* in Eqs. (1.1) and (1.2) are continuous and T-periodic in the time t; moreover, the nonlinear functions F_j, G_j, G_j^* are analytic in the argument x, the small parameter μ, the function u_j, and their first-order partial derivatives u_{jx}, u_{jt} with respect to x and t. Here and below the plus sign in the gyroscopic terms with double subscripts applies for $j = 1$ and the minus sign for $j = 2$. The coefficients ρ, σ, $\varkappa$, $\varkappa_1 \varkappa_2$, c, K_1, K_0, m are constant positive quantities. The nonlinear functions F_j, G_j and G_j^* can depend in individual cases on partial derivatives of higher than the first order, but this merely complicates the expressions without introducing any additional difficulties. The permissible slight asymmetry of the gyro system is allowed for by the small linear terms referred to the functions F_j, G_j and G_j^*.

Equations of the type (1.1) and (1.2) describe, among other things, the motion of ultracentrifuges, flexible rotors with attached masses, etc.

Let us assume that boundary value problem (1.1), (1.2) has a periodic solution $u_j^\circ(x, t)$ which can be determined either in exact (closed) form or by one of the approximate methods. We are to investigate the stability of this solution.

Setting $\xi_j = u_j - u_j^\circ$, we find that functions ξ_j must satisfy the equations in variations

$$\frac{\partial^4 \xi_j}{\partial x^4} + \rho \frac{\partial^2 \xi_j}{\partial t^2} + \mu\varkappa \frac{\partial \xi_j}{\partial t} = \mu P_\xi [F_j] \qquad (1.3)$$

where

$$\xi_j = \partial \xi_j / \partial x = 0$$

for $x = 0$, and

$$\sigma \frac{\partial^2 \xi_j}{\partial x^2} = -K_1 \frac{\partial^3 \xi_j}{\partial x \partial t^2} + i^{2j} K_0 \omega \frac{\partial^2 \xi_{j\pm 1}}{\partial x \partial t} - c \frac{\partial \xi_j}{\partial x} - \mu\varkappa_1 \frac{\partial^2 \xi_j}{\partial x \partial t} + \mu P_\xi [G_j]$$

$$\sigma \frac{\partial^3 \xi_j}{\partial x^3} = m \frac{\partial^2 \xi_j}{\partial t^2} + \mu\varkappa_2 \frac{\partial \xi_j}{\partial t} + \mu P_\xi [G_j^*] \qquad (1.4)$$

for $x = l$. The homogeneous linear operator P [ ] is given by the expression

$$P_\xi [\ ] = \sum_{\beta=1,2} \left\{ \xi_\beta \left(\frac{\partial}{\partial u_\beta} \right) + \frac{\partial \xi_\beta}{\partial x} \left(\frac{\partial}{\partial u_{\beta x}} \right) + \frac{\partial \xi_\beta}{\partial t} \left(\frac{\partial}{\partial u_{\beta t}} \right) \right\} [\ ] \qquad (1.5)$$

The parentheses in (1. 5) mean that the derivatives are computed for unperturbed motion.

For $\mu = 0$ the natural frequencies and modes of the bending oscillations of a linear gyroscopic system can be found by setting

$$\xi_1 = Y_1 (x, \lambda) e^{i\lambda t}, \quad \xi_2 = iY_2 (x, \lambda) e^{i\lambda t} \qquad (1.6)$$

The solutions of Eqs. (1. 3) for $\mu = 0$ and $Y_j (0, \lambda) = Y_j' (0, \lambda) = 0$ are

$$Y_j (x, \lambda) = C_j U (kx) + D_j V (kx), \ k^4 = \rho\lambda^2 \ (j = 1, 2) \qquad (1.7)$$

where C_j and D_j are arbitrary constants and $U (kx)$, $V (kx)$ are Krylov functions. For $\mu = 0$ expression (1. 4) yields a system of homogeneous linear equations in the arbitrary constants C_j and D_j,

$$a_{11}C_1 + a_{12}D_1 - a_{13}C_2 - a_{14}D_2 = 0, \quad a_{21}C_1 + a_{22}D_1 = 0$$
$$-a_{13}C_1 - a_{14}D_1 + a_{11}C_2 + a_{12}D_2 = 0, \quad a_{21}C_2 + a_{22}D_2 = 0 \qquad (1.8)$$

$$a_{11} = \sigma k^2 S - kT (K_1\lambda^2 - c), \ a_{21} = \sigma k^3 V + m\lambda^2 U, \quad a_{13} = K_0 \omega\lambda kT$$
$$a_{12} = \sigma k^2 T - kU (K_1\lambda^2 - c), \quad a_{22} = \sigma k^3 S + m\lambda^2 V, \quad a_{14} = K_0 \omega\lambda kU \qquad (1.9)$$

Here and below the Krylov functions S, T, U, V are computed for the argument kl, which we omit for the sake of simplicity of notation (*). The natural frequencies can be determined from the condition of equality to zero of the determinant of system (1. 8),

$$\begin{vmatrix} a_{11} & a_{12} & -a_{13} & -a_{14} \\ -a_{13} & -a_{14} & a_{11} & a_{12} \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} \end{vmatrix} = 0 \qquad (1.10)$$

which yields that two transcendental frequency equations

$$\Delta_1 = a_{11}a_{22} - a_{12}a_{21} + a_{22}a_{13} - a_{21}a_{14} = 0 \qquad (1.11)$$
$$\Delta_2 = a_{11}a_{22} - a_{12}a_{21} - a_{22}a_{13} + a_{21}a_{14} = 0 \qquad (1.12)$$

or, in expanded form

$$(K_1\lambda^2 - c) [\sigma k^3 (UV - ST) + m\lambda^2 (U^2 - VT)] + \sigma\lambda^2 [\rho (S^2 - TV) +$$
$$+ mk (SV - UT) \pm K_0 \omega\lambda k [\sigma k^3 (ST - UV) + m\lambda^2 (TV - U^2)] = 0$$

It is clear that all the roots of Eq. (1. 10) are real numbers, and that the positive roots of Eq. (1. 11) are equal to the negative roots of Eq. (1. 12), and vice versa. We infer from (1. 8) that

$$\frac{D_1}{C_1} = \frac{D_2}{C_2} = -\frac{a_{21}}{a_{22}}, \qquad C_2 = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{13}a_{22} - a_{14}a_{21}} C_1$$

so that, by (1. 7), $Y_1 = -Y_2 = Y^*$ for every root of Eq. (1. 11) and $Y_1 = Y_2 = Y^*$ for every root of Eq. (1. 12), where

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*) We recall that the Krylov functions of transverse oscillations are

$$S(\alpha) = 0.5 (\text{ch}\alpha + \cos\alpha), \qquad T(\alpha) = 0.5 (\text{sh}\alpha + \sin\alpha)$$
$$U(\alpha) = 0.5(\text{ch}\alpha - \cos\alpha), \qquad V(\alpha) = 0.5(\text{sh}\alpha - \sin\alpha)$$

$$Y^*(x,\lambda) = U(kx) - a_{21}a_{22}^{-1}V(kx) \tag{1.13}$$

to within a constant factor.

Substituting the values of $Y_j(x, \lambda)$ into (1.6) and separating the real parts for the imaginary, we see that the real roots of Eq. (1.11) or the negative roots of Eq. (1.12) correspond to forward precession of the gyro system axes, and the negative roots of Eq. (1.11) or the positive roots of Eq. (1.12). to its reverse precession.

Let us denote one of the natural frequencies of the linearized gyro system which is not a multiple of $2\pi / T$ by λ_s, and let us assume (to be specific) that λ_s is a simple root of Eq. (1.11). In order to find the characteristic exponent α close to $i\lambda_s$ of equations in variations (1.3) and (1.4), we set

$$\xi_1 = y_1(\mu, x, t)\, e^{\alpha t}, \qquad \xi_2 = iy_2(\mu, x, t)\, e^{\alpha t} \tag{1.14}$$

where $y_j(\mu, x, t)$, $j = 1, 2$, must be T-periodic functions; we then seek α and $y_j(\mu, x, t)$ in series form,

$$\alpha = i\lambda_s + \mu a + \mu^2 a_2 + \ldots, \quad y_j(\mu, x, t) = y_j^\circ(x, t) + \mu y_j^{(1)} + \ldots \tag{1.15}$$

Equations (1.3) and (1.4) imply that the functions $y_j^\circ(x, t)$ must satisfy the differential equations

$$\frac{\partial^4 y_j^\circ}{\partial x^4} + \rho A[y_j^\circ] = 0 \tag{1.16}$$

where

$$y_j^\circ = \partial y_j^\circ / \partial x = 0$$

for $x = 0$, and

$$B_1[y_j^\circ] - K_0\omega\left(\lambda_s \frac{\partial \overset{\circ}{y}_{j\pm 1}}{\partial x} - i\,\frac{\partial^2 \overset{\circ}{y}_{j\pm 1}}{\partial x \partial t}\right) = 0, \qquad B_2[y_j^\circ] = 0 \tag{1.17}$$

for $x = l$. The same equations also imply that the functions $y_j^{(1)}(x, t)$ must satisfy the differential equations

$$\frac{\partial^4 y_j^{(1)}}{\partial x^4} + \rho A[y_j^{(1)}] = -(2a\rho + \varkappa)\left(i\lambda_s y_j^\circ + \frac{\partial y_j^\circ}{\partial t}\right) + i^{1-j}R[F_j] \tag{1.18}$$

and the boundary conditions

$$B_1[y_j^{(1)}] - K_0\omega\left(\lambda_s \frac{\partial y_{j\pm 1}^{(1)}}{\partial x} - i\,\frac{\partial^2 y_{j\pm 1}^{(1)}}{\partial x \partial t}\right) =$$
$$= -(2aK_1 + \varkappa_1)\left(i\lambda_s \frac{\partial y_j^\circ}{\partial x} + \frac{\partial^2 y_j^\circ}{\partial x \partial t}\right) - iK_0\omega a\,\frac{\partial \overset{\circ}{y}_{j\pm 1}}{\partial x} + i^{1-j}R[G_j]$$
$$B_2[y_j^{(1)}] = i^{1-j}R[G_j^*] + (2am + \varkappa_2)\left(i\lambda_s y_j^\circ + \frac{\partial y_j^\circ}{\partial t}\right) \tag{1.19}$$

The linear operators in Eqs. (1.16)–(1.19) are given by the formulas

$$A[\ ] = \left(-\lambda_s^2 + 2i\lambda_s \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\right)[\ ]$$
$$B_1[\ ] = \sigma \frac{\partial^2}{\partial x^2}[\ ] + K_1 \frac{\partial}{\partial x} A[\ ] + c\frac{\partial}{\partial x}[\ ]$$
$$B_2[\ ] = \sigma \frac{\partial^3}{\partial x^3}[\ ] - mA[\ ] \tag{1.20}$$
$$R[\ ] = \sum_{\beta=1,2} i^{\beta-1}\left\{y_\beta^\circ\left(\frac{\partial}{\partial u_\beta}\right) + \frac{\partial y_\beta^\circ}{\partial x}\left(\frac{\partial}{\partial u_{\beta x}}\right) + \left(i\lambda_s y_\beta^\circ + \frac{\partial y_\beta^\circ}{\partial t}\right)\left(\frac{\partial}{\partial u_{\beta t}}\right)\right\}[\ ]$$

Let us suppose that in addition to the root λ_s, Eq. (1.10) also has simple roots of the form $\lambda_{sr} = \lambda_s + n_r \omega$, $r = 0, 1, \ldots, \gamma$, where n_r are positive or negative integers, where the first r' roots $\lambda_{s1}, \ldots, \lambda_{sr'}$ belong to Eq. (1.11) and the remaining $\gamma - r'$ roots $\lambda_{s,r'+1}, \ldots, \eta_{s\gamma}$ belong to Eq. (1.12). The boundary value problem for the functions $y_j^\circ(x, t)$, $j = 1, 2$ then has T-periodic solutions containing $\gamma + 1$ arbitrary constants M_r,

$$y_1^\circ = \sum_{r=0}^{r'} M_r e^{in_r\omega t} Y_{sr}(x) + \sum_{r=r'+1}^{\gamma} M_r e^{in_r\omega t} Y_{sr}(x)$$
$$y_2^\circ = -\sum_{r=0}^{r'} M_r e^{in_r\omega t} Y_{sr}(x) + \sum_{r=r'+1}^{\gamma} M_r e^{in_r\omega t} Y_{sr}(x) \tag{1.21}$$

Here $Y_{sr}(x)$ are the eigenfunctions corresponding to the roots λ_{sr} and defined by formula (1.13), where $n_0 = 0$ and $\lambda_{s0} = \lambda_s$. Each of the functions $y_j^{(1)}(x, t)$ is obtainable as a sum of T-periodic functions,

$$y_j^{(1)}(x, t) = \sum_{r=0}^{\gamma} z_{jr}(x) e^{in_r\omega t} + z_j(x, t) \tag{1.22}$$

By virtue of (1.18), (1.19) and (1.21) the functions $z_{jr}(x)$ must satisfy the ordinary differential equations

$$z_{jr}^{IV} - \rho\lambda_{sr}^2 z_{jr} = i^{1-j} p_{jn_r} \begin{cases} + i^{2j+1}(2a\rho + \varkappa) Y_{sr}\lambda_{sr} M_r & (r < r') \\ - i(2a\rho + \varkappa) Y_{sr}\lambda_{sr} M_r & (r > r') \end{cases} \tag{1.23}$$

where

$$z_{jr} = z_{jr}' = 0$$

for $x = 0$, and

$$\sigma z_{jr}'' - (K_1\lambda_{sr}^2 - c) z_{jr}' - K_0\omega\lambda_{sr} z_{j\pm1}' =$$
$$= i^{1-j} q_{jn_r}(l) \begin{cases} + i^{2j+1} M_r Y_{sr}' [a(2K_1\lambda_{sr} - K_0\omega) + \varkappa_1\lambda_{sr}] & (r < r') \\ - i M_r Y_{sr}' [a(2K_1\lambda_{sr} + K_0\omega) + \varkappa_1\lambda_{sr}] & (r > r') \end{cases}$$

$$\sigma z_{jr}''' + m\lambda_{sr}^2 z_{jr} = i^{1-j} q_{jn_r}(l) \begin{cases} - i^{2j+1}(2am + \varkappa_2)\lambda_{sr} Y_{sr} M_r & (r < r') \\ + i(2am + \varkappa_2)\lambda_{sr} Y_{sr} M_r & (r > r') \end{cases} \quad (j = 1, 2) \tag{1.24}$$

for $x = l$; the functions $z_j(x, t)$ must satisfy the equations

$$\frac{\partial^4 z_j}{\partial x^4} + \rho A[z_j] = i^{1-j} \sum_{n=-\infty}^{\infty}{}' p_{jn}(x) e^{in\omega t} \tag{1.25}$$

where

$$z_j = \partial z_j / \partial x = 0$$

for $x = 0$, and

$$B_1[z_j] - K_0\omega\left(\lambda_s \frac{\partial z_{j\pm1}}{\partial x} - i\frac{\partial^2 z_{j\pm1}}{\partial x \partial t}\right) = i^{1-j} \sum_{n=-\infty}^{\infty}{}' q_{jn}(l) e^{in\omega t}$$
$$B_2[z_j] = i^{1-j} \sum_{n=-\infty}^{\infty}{}' q_{jn}^*(l) e^{in\omega t} \tag{1.26}$$

for $x = l$.

The primes next to the summation symbols mean that $n \neq n_r$.

In Eqs. (1.23)–(1.26) $p_{jn}(x)$, $q_{jn}(x)$, $q_{jn}^*(x)$ are the coefficients of the Fourier series of the functions $R[F_j]$, $R[G_j]$ and $R[G_j^*]$, respectively, where the operator

$R[\ ]$ (by virtue of (1.20) and (1.21)) is given by the formula

$$R[\ ] = \sum_{r=0}^{r'} M_r e^{in_r\omega t}\left\{Y_{sr}\left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right) + Y_{sr}'\left(\frac{\partial}{\partial u_{1x}} - i\frac{\partial}{\partial u_{2x}}\right) + \right. \tag{1.27}$$

$$\left. + i\lambda_{sr}Y_{sr}\left(\frac{\partial}{\partial u_{1t}} - i\frac{\partial}{\partial u_{2t}}\right)\right\}[\ ] + \sum_{r=r'+1}^{\Upsilon} M_r e^{in_r\omega t}\left\{Y_{sr}\left(\frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2}\right) + \right.$$

$$\left. + Y_{sr}'\left(\frac{\partial}{\partial u_{1x}} + i\frac{\partial}{\partial u_{2x}}\right) + i\lambda_{sr}Y_{sr}\left(\frac{\partial}{\partial u_{1t}} + i\frac{\partial}{\partial u_{2t}}\right)\right\}[\ ]$$

All of the partial derivatives of the functions F_j, G_j and C_j^* must be computed for unperturbed motion. We infer from (1.21) that the Fourier coefficients $p_{jn}(x)$, $q_{jn}(x)$ and $q_{jn}^*(x)$ are homogeneous linear functions of the constants M_r. Specifically,

$$p_{jn_r}(x) = \frac{1}{T}\int_0^T R[F_j]\, e^{-in_r\omega t}dt = \sum_{k=0}^{\Upsilon} M_k p_{jrk}(x)$$

$$q_{jn_r}(x) = \sum_{k=0}^{\Upsilon} M_k q_{jrk}(x), \qquad q_{jn_r}^*(x) = \sum_{k=0}^{\Upsilon} M_k q_{jrk}^*(x)$$

The solutions of Eqs. (1.23) satisfying the boundary conditions for $x = 0$ are

$$z_{jr}(x) = C_{jr}U(k_r x) + D_{jr}V(k_r x) + i^{2j+1}M_r\Psi_r(x)(2a\rho + \varkappa) +$$

$$+ i^{1-j}\sum_{\delta=0}^{\Upsilon} M_k \Psi_{jr\delta}(x) \quad (k_r^4 = \rho\lambda_{sr}^2)$$

where C_{jr} and D_{jr} are arbitrary constants, and

$$\Psi_r(x) = \frac{k_r}{\rho\lambda_{sr}}\int_0^x V[k_r(x-\zeta)\, Y_{sr}(\zeta)\, d\zeta]$$

$$\Psi_{jr\delta}(x) = \frac{1}{k_r^3}\int_0^x V[k_r(x-\zeta)\, p_{jr\delta}(\zeta)\, d\zeta] \tag{1.28}$$

Boundary conditions (1.24) yield the system (1.29)

$$a_{11}C_{1r} + a_{12}D_{1r} - a_{13}C_{2r} - a_{14}D_{2r} = b_{1r}(a), \qquad a_{21}C_{1r} + a_{22}D_{1r} = b_{3r}(a)$$

$$- a_{13}C_{1r} - a_{14}D_{1r} + a_{11}C_{2r} + a_{12}D_{2r} = b_{2r}(a), \qquad a_{21}C_{2r} + a_{22}D_{2r} = b_{4r}(a)$$

Here the coefficients a_{ij} are given by formulas (1.9) for $\lambda = \lambda_{sr}$, and b_{ir} are linear functions of the required quantity a given by

$$b_{jr} = i^{2j+1}M_r\{(2a\rho + \varkappa)[(K_1\lambda_{sr}^2 - K_0\omega\lambda_{sr} - c)\Psi_r'(l) - \sigma\Psi_r''(l)] + \tag{1.30}$$

$$+ Y_{sr}'(l)[a(2K_1\lambda_{sr} - K_0\omega) + \varkappa_1\lambda_{sr}]\} + i^{1-j}\sum_{h=0}^{\Upsilon} M_h d_{jrh}$$

$$b_{j+2,r} = -i^{2j+1}M_r\{(2a\rho + \varkappa)[\sigma\Psi_r'''(l) + m\lambda_{sr}^2\Psi_r(l)] + (2am +$$

$$+ \varkappa_2)\lambda_{sr}Y_{sr}(l)\} + i^{1-j}\sum_{h=0}^{\Upsilon} M_h d_{jrh}^*$$

for $r < r'$, and

$$b_{jr} = -iM_r\{(2a\rho + \varkappa)[(K_1\lambda_{sr}^2 + K_0\omega\lambda_{sr} - c)\Psi_r'(l) - \sigma\Psi_r''(l)] + Y_{sr}'(l)[a(2K_1\lambda_{sr} + K_0\omega) + \varkappa_1\lambda_{sr}]\} + i^{1-j}\sum_{h=0}^{\gamma} M_h d_{jrh} \qquad (1.31)$$

$$b_{j+2,r} = iM_r\{(2a\rho + \varkappa)[\sigma\Psi_r'''(l) + m\lambda_{sr}^2\Psi_r(l)] + (2am + \varkappa_2)\lambda_{sr}Y_{sr}(l)\} + i^{1-j}\sum_{h=0}^{\gamma} M_h d_{jrh}^* \qquad (j = 1, 2)$$

for $r > r'$.

In formulas (1.30) and (1.31) we have

$$d_{jrh} = \Psi_{jrh}'(l)(K_1\lambda_{sr}^2 - c) - \sigma\Psi_{jrh}''(l) + q_{jrh}(l) - i^{2j-1}K_0\omega\lambda_{sr}\Psi_{j\pm1,rh}(l)$$

$$d_{jrh}^* = q_{jrh}^*(l) - \sigma\Psi_{jrh}'''(l) - m\lambda_{sr}^2\Psi_{jrh}(l)$$

Since λ_{sr} is a root of Eq. (1.10), it follows that the determinant of system (1.29) is equal to zero, and that Eqs. (1.29) are compatible if

$$\begin{vmatrix} a_{11} & a_{12} & -a_{13} & b_{1r} \\ -a_{13} & -a_{14} & a_{11} & b_{2r} \\ a_{21} & a_{22} & 0 & b_{3r} \\ 0 & 0 & a_{21} & b_{4r} \end{vmatrix} = 0 \qquad (1.32)$$

or

$$P_r = B_{r0}M_0 + B_{r1}M_1 + \ldots + (B_{rr} - aA_r)M_r + \ldots + B_{r\gamma}M_\gamma = 0 \qquad (1.33)$$

where B_{ri} and A_r $(r, i = 0, 1, \ldots, \gamma)$ are known complex numbers. Hence, the quantity a must be a root of the algebraic equation

$$\begin{vmatrix} B_{00} - aA_0 & B_{01} & \ldots & B_{0\gamma} \\ B_{10} & B_{11} - aA_1 & \ldots & B_{1\gamma} \\ \ldots & \ldots & \ldots & \ldots \\ B_{\gamma 0} & B_{\gamma 1} & \ldots & B_{\gamma\gamma} - aA_\gamma \end{vmatrix} = 0 \qquad (1.34)$$

Let us suppose that Eq. (1.34) has simple roots only. Then, as we know, it is possible to obtain arbitrarily exact expressions for all $\gamma + 1$ characteristic exponents corresponding to the proper values λ_{sr} of generating system (1.3), (1.4). However, the stability of the unperturbed periodic solution can usually be established on the basis of the first approximation. Expression (1.33) yields the quantities M_r for every simple root a of Eq. (1.34); one of these quantities, let us say M_γ, can be chosen arbitrarily, and

$$\frac{\partial(P_0, \ldots, P_\gamma)}{\partial(M_0, \ldots, M_{\gamma-1}, a)} \neq 0$$

after which boundary value problem (1.25), (1.26) defines the T-periodic function $z_j(x, t)$ to within a constant, since the frequencies $n\omega$ are not natural frequencies for $n \neq n_r$.

In a special case Eq. (1.10) may not have roots which differ from λ_s by quantities which are multiples of ω; this simplifies the determination of the coefficient a in expansion (1.15) of the characteristic exponent close to $i\lambda_s$. The periodic functions y_j^0 are equal to $y_1^0 = -y_2^0 = Y_s$ instead of being given by (1.21), and each of the functions $y_j^{(1)}(x, t)$ can be expressed as a sum of two periodic functions

$y_j^{(1)}(x, t) = Z_j(x) + z_j(x, t)$, one of which, $Z_j(x)$, does not depend on time, while the average value of the other, $z_j(x, t)$, is equal to zero. It is easy to show that the quantity a must satisfy linear equation (1.32) in which the elements of the last column are given by

$$b_j = i^{2j+1}\{(2a\rho + \varkappa)[(K_1\lambda_s^2 - K_0\omega\lambda_s - c)\Psi'(l) - \sigma\Psi''(l)] + Y_s'(l)[a(2\lambda_s K_1 - K_0\omega) + \varkappa_1\lambda_s]\} + i^{1-j}\left\{(K_1\lambda_s^2 - c)\Psi_j'(l) - \sigma\Psi_j''(l) + \frac{1}{T}\int_0^T R[G_j]\right\} - i^j K_0\omega\lambda_s\Psi'_{j\pm1}(l)$$

$$b_{j+2} = -i^{2j+1}\{(2a\rho + \varkappa)[\sigma\Psi'''(l) + m\lambda_s^2\Psi(l)] + (2am + \varkappa_2)\lambda_s Y_s\} + i^{1-j}\left\{\frac{1}{T}\int_0^T R[G_j^*]\,dt - m\lambda_s^2\Psi_j(l) - \sigma\Psi_j'''(l)\right\} \quad (1.35)$$

$$\Psi(x) = \frac{k_s}{\rho\lambda_s}\int_0^x V[k(x-\zeta)]\,Y(\zeta)\,d\zeta$$

$$\Psi_j(x) = \frac{k_s}{\rho\lambda_s^2 T}\int_0^x\int_0^T V[k(x-\zeta)R[F_j]\,dt\,d\zeta]$$

The operator $R\ [\ ]$ is given by

$$R[\ ] = \left\{Y_s(x)\left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2}\right) + Y_s'(x)\left(\frac{\partial}{\partial u_{1x}} - i\frac{\partial}{\partial u_{2x}}\right) + i\lambda_s Y_s\left(\frac{\partial}{\partial u_{1t}} - i\frac{\partial}{\partial u_{2t}}\right)\right\}[\ ]$$

instead of (1.27).

The roots of Eqs. (1.10) may include resonance roots given by $\lambda_v = \pm n_v\,\omega$, $v = 1, \ldots, \beta$, where n_v are integers whose signs are chosen in such a way that $\lambda_v = n_v\,\omega$ are the roots of Eq. (1.11), and $-\lambda_v$ are the roots of Eq. (1.12). In the resonance case the characteristic exponents must be computed in the form of the series $\alpha = \mu a + \mu^2 a_2 + \ldots$ rather then in the form (1.15); this means that Eqs. (1.16)–(1.20) all remain valid if we set $\lambda_s = 0$.

The boundary value problem for the functions y_j^0 in the resonance case has a periodic solution which depends on 2β arbitrary constants M,

$$y_1^\circ = \sum_{v=1}^{\beta} Y_v(x, \lambda_v)\,(M_v e^{i\lambda_v t} + M_{v+\beta}e^{-i\lambda_v t})$$

$$y_2^\circ = \sum_{v=1}^{\beta} Y_v(x, \lambda_v)\,(-M_v e^{i\lambda_v t} + M_{v+\beta}e^{-i\lambda_v t})$$

The eigenfunctions $Y_v(x, \lambda_v)$ in this expression can be determined from formula (1.13) by setting $\lambda = \lambda_v = n_v\omega$. We can express each function $y_j^{(1)}(x, t)$ as a sum of $2\pi/\omega$-periodic functions,

$$y_j^{(1)}(x, t) = \sum_{v=1}^{\beta}[z_{jv}(x)\,e^{i\lambda_v t} + z_{j,v+\beta}(x)\,e^{-i\lambda_v t}] + z_j(x, t)$$

The subsequent argument is the same as that above and need not be repeated. We merely note that the quantity a and the constants $M_1, \ldots, M_{2\beta}$, one of which can be chosen arbitrarily, must satisfy 2β Eqs. (1.32); the coefficients b_{ir} in these equations have the same structure as (1.30), (1.31), where the operator $R[\ ]$ in the resonance case is given by the formula

$$R[\ ] = \sum_{v=1}^{\beta} e^{i\lambda_v t} M_v \left\{ Y_v \left(\frac{\partial}{\partial u_1} - i\frac{\partial}{\partial u_2} \right) + Y_v' \left(\frac{\partial}{\partial u_{1x}} - i\frac{\partial}{\partial u_{2x}} \right) + \right.$$

$$\left. + i\lambda_v Y_v \left(\frac{\partial}{\partial u_{1t}} - i\frac{\partial}{\partial u_{2t}} \right) \right\} [\ ] + \sum_{v=1}^{\beta} e^{-i\lambda_v t} M_{v+\beta} \left\{ Y_v \left(\frac{\partial}{\partial u_1} + i\frac{\partial}{\partial u_2} \right) + \right.$$

$$\left. + Y_v' \left(\frac{\partial}{\partial u_{1x}} + i\frac{\partial}{\partial u_{2x}} \right) - i\lambda_v Y_v \left(\frac{\partial}{\partial u_{1t}} + i\frac{\partial}{\partial u_{2t}} \right) \right\} [\ ]$$

Note. Elastic gyroscopic systems can be more complex than those considered here; specifically, they may contain discrete perameters not only at the ends of the interval $0 \leqslant x \leqslant l$, but also at several points inside it; the matching conditions at the boundaries between domains and the dispositions of the latter can be quasilinear. To investigate the stability of the periodic solutions of such gyro systems we must introduce matrices of the discrete parameters and elastic domains between them. Multiplying these matrices, we obtain equations of the form (1.34) for the quantity a in expansion (1.15) of the characteristic exponent.

2. As already noted, the proposed theory of stability of the periodic solutions of quasilinear gyro systems can be used, among other things, for studying the dynamics of flexible rotors. The nonlinear functions F_j, G_j, G_j^* in equations of motion (1.1), (1.2) are general enough to enable us to investigate the stability of periodic oscillations of flexible rotors or their relative equilibrium state under the action of numerous factors which can give rise to almost-periodic autooscillatory states. As we know, these factors include internal and external friction, asymmetry with respect to rigidity and to the moments of inertia, various hydraulic forces, etc. In addition, the nonlinear equations of motion of the rotors can have several periodic solutions, and it is extremely important to determine which of them are stable.

As an example let us consider one such problem which has received almost no attention in the literature despite its considerable practical importance.

An imperfectly balanced flexible rotor rests on several isotropic elastic supports, some of which have nonlinear characteristics. These characteristics are sometimes introduced intentionally (e.g. in ultracentrifuges) in order to prevent hazardous vibrations of the rotor over a broad range of angular velocities (in which case the supports in question act as nonlinear dampers); in other cases the supports are nonlinear for technological reasons. We assume that the system is quite complex. In addition to the distributed mass of the rotor we must also take into account the elements attached to it (the longer ones can be fastened to the rotor at two or more places). We are to determine the possible periodic variation states of such a system and investigate their stability. For simplicity we shall take into account only the nonlinearity of the supports themselves, neglecting the other forms of nonlinearity; in addition, we regard the moving parts of the supports and the elements in direct contact with them as bodies of small dimensions, arbitrarily referring to them as "disks".

Let s be the number of nonlinear supports and $l_1 < l_2 \ldots < l_s$ the abscissas of their locations along the length of the rotor. We introduce the complex deflection $w(x, t) = u_1(x, t) + iu_2(x, t)$, where $u_j(x, t)$ are the projections of the deflection of the axial line of the rotor on the fixed coordinate planes xy and xz. The function $w(x, t)$ must satisfy

1) the following differential equation in each domain:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho\frac{\partial^2 w}{\partial t^2} + \varkappa\frac{\partial w}{\partial t} = \rho\omega^2\varepsilon(x)\exp i[\omega t + \theta(x)] \tag{2.1}$$

2) the following nonlinear matching conditions for $x = l_r$ $(r = 1,\ldots,s)$:

$$w(l_r + 0) = w(l_r - 0) = w_r, \qquad \frac{\partial w(l_r + 0)}{\partial x} = \frac{\partial w(l_r - 0)}{\partial x} = \frac{\partial w_r}{\partial x}$$

$$EI\left[\frac{\partial^2 w(l_r + 0)}{\partial x^2} - \frac{\partial^2 w(l_r - 0)}{\partial x^2}\right] = K_{1r}\frac{\partial^3 w_r}{\partial x\partial t^2} - i\omega K_{0r}\frac{\partial^2 w_r}{\partial x\partial t} + \varkappa_{1r}\frac{\partial^2 w_r}{\partial x\partial t} + L_r\left(\left|\frac{\partial w_r}{\partial x}\right|\right)\frac{\partial w_r}{\partial x}\left|\frac{\partial w_r}{\partial x}\right|^{-1}$$

$$EI\left[\frac{\partial^3 w_r(l_r + 0)}{\partial x^3} - \frac{\partial^3 w(l_r - 0)}{\partial x^3}\right] = -m_r\frac{\partial^2 w_r}{\partial t^2} - \varkappa_{2r}\frac{\partial w}{\partial t} - Q_r(|w_r|)w_r|w_r|^{-1} + m_r\varepsilon_r\omega^2\exp i(\omega t + \theta_r) \tag{2.2}$$

3) four boundary conditions which depend on the mode of support of the rotor ends;

4) various linear matching conditions at the sites of the rigid or elasticomassive supports with a linear characteristic, concentrated masses, etc.

In Eqs. (2.1), (2.2) $l_r \pm 0$ are the abscissas of points on the rotor axis to the right and to the left of the point $x = l_r$ and infinitely close to it; $\varepsilon(x)$, ε_r are the eccentricities along the rotor and disk; $\theta(x)$, θ_r are the angles between the eccentricity vectors and some plane rotating together with the rotor at the angular velocity ω; m_r is the mass of the disk; K_{0r}, K_{1r} are its polar and equatorial moments of inertia; EI and ρ are the constant bending rigidity of the rotor and its mass per unit length; $\varkappa$, $\varkappa_{1r}$ and $\varkappa_{2r}$ are the damping factors. The reaction of the support is directed opposite to the complex deflection w_r: its magnitude is a nonlinear function $Q_r(|w_r|)$ of its absolute value. Similarly, the vector of the moment in the support is opposite to the complex-angle vector $\partial w_r/\partial x$; its magnitude is a nonlinear function $L_r(|\partial w_r/\partial x|)$ of its absolute value. For $m_r = K_{1r} = K_{0r} = 0$ we obtain the conditions of matching without allowance for the mass of the nonlinear support itself.

The linear matching conditions (which include the boundary conditions) will not be written out. The simplest of them can be obtained from (2.2). Thus, in the case of elastic supports with a linear characteristic $Q = c|w|$ and $L = d|\partial w/\partial x|$, where c and d are constants; if there is a disk but no support at the boundary between two adjacent domains, then $Q = L = 0$, etc.

Setting $w(x, t) = W(x)e^{i\omega t}$ in (2.1) and (2.2), we obtain the differential equation

$$W^{\mathrm{IV}}(x) - k^4 W(x) = \frac{\rho\omega^2}{EI}\varepsilon(x)e^{i\theta(x)}, \qquad k = \left(\frac{\rho\omega^2}{EI}\right)^{1/4}\left(1 - \frac{i\varkappa}{4\omega\rho}\right) \tag{2.3}$$

for the function $W(x)$ and the following nonlinear matching conditions for $x = l_r$:

$$W(l_r + 0) = W(l_r - 0) = W_r, \qquad W'(l_r + 0) = W'(l_r - 0) = W_r'$$

$$EI[W''(l_r + 0) - W''(l_r - 0)] = [\omega^2 K_r + i\omega\varkappa_{1r} + L_r(|W_r'|)|W_r'|^{-1}]W_r'$$

$$EI\,[W'''(l_r+0)-W'''(l_r-0)]=[m_r\omega^2-i\omega\varkappa_{2r}-Q_r(|W_r|)|W_r|^{-1}]\,W_r+m_r\varepsilon_r\omega^2e^{i\theta_r}$$
$$(K_r=K_{0r}-K_{1r}) \qquad (2.4)$$

where the primes denote the derivatives of the function W with respect to x.

Let us denote by $X(x)$ a vector (column matrix) whose components are equal to $\{W(x),\ W'(x)\ EIW''(x),\ EIW'''(x)\}$. Equations (2.3) and all the matching conditions are linear over all the intervals $0\leqslant x<l_1,\ldots,l_r<x<l_{r+1}$ $(r=1,\ldots,s)$; the vectors X_{l_r+0} and $X_{l_{r+1}-0}$ are therefore related by the matrix equation

$$X_{l_{r+1}-0}=\|p_{ki}^{(r)}(\omega)\|\,X_{l_r+0}+\|p_k^{(r)}(\omega)\| \qquad (2.5)$$

where the vector $\|p_k^{(r)}(\omega)\|$ (4×1) and the fourth-order square matrix $\|p_{ki}^{(r)}(\omega)\|$ can be determined by the standard methods of the theory of linear oscillations; specifically, the matrix $\|p_{ki}^{(r)}(\omega)\|$ is a product of the matrices of the elastic domains, discrete masses, linear supports, etc., over the interval $l_r<x<l_{r+1}$.

The basic unknowns are the complex deflections W_r and angles of rotation W_r' of the rotor cross sections over the nonlinear supports $(r=1,\ldots,s)$. Satisfying Eqs. (2.3), the boundary conditions at the left end of the rotor and all the linear matching conditions over the interval $0<x<l_1$, and then eliminating the initial parameters, we obtain the following expressions for $x=l_1-0$:

$$EIW_1''=\alpha_{31}W_1+\beta_{31}W_1'+\alpha_3, \qquad EIW_1'''=\alpha_{41}W_1+\beta_{42}W_1'+\alpha_4$$

where α_{ki}, β_{ki} and α_k are known functions of the angular velocity ω. By virtue of (2.4), we can now express the vector $X(x)$ for $x=l_1+0$ as a sum of the product of the coagulated matrices and the vector H ,

$$X_{l_1+0}=\left\|\begin{matrix}E_2\\ g_1\end{matrix}\right\|\times\|h_1\|+H_{l_1+0} \qquad (2.6)$$

where E_2 is a second-order identity matrix,

$$g_1=\left\|\begin{matrix}\alpha_{31}, & \beta_{31}+\omega^2K_1+i\omega\varkappa_{11}+L_1(|W_1'|)|W_1'|^{-1}\\ \alpha_{41}+m_1\omega^2-i\omega\varkappa_{21}-Q_1(|W_1|)|W_1|^{-1}, & \beta_{41}\end{matrix}\right\|, \qquad h_1=\left\|\begin{matrix}W_1\\ W_1'\end{matrix}\right\|$$

$$H_{l_1+0}=\{0,0,\alpha_3,\alpha_4+m_1\varepsilon_1\omega^2\exp i\theta_1\}$$

Next, we infer from (2.5) and (2.6) that

$$X_{l_2-0}=\Lambda_1^{(2)}W_1+\Xi_1^{(2)}W_1'+H_{l_2-0}$$

Here the vectors $\Lambda_1^{(2)}$, $\Xi_1^{(2)}$ and H_{l_2-0} are given by

$$\Lambda_1^{(2)}=\|\alpha_{k1}^{(2)}+\gamma_{k1}^{(2)}Q_1(|W_1|)|W_1|^{-1}\|$$
$$\Xi_1^{(2)}=\|\beta_{k1}^{(2)}+\delta_{k1}^{(2)}L_1(|W_1'|)|W_1'|^{-1}\| \qquad H_{l_2-0}=\|\alpha_k^{(2)}\|$$

where $k=1,2,3,4$ are the row numbers.

Proceeding from left to right and carrying out the same operations, we can readily show that the vector

$$X_{l_r-0}=\sum_{\nu=1}^{r-1}(\Lambda_\nu^{(r)}W_\nu+\Xi_\nu^{(r)}W_\nu')+H_{l_r-0} \qquad (2.7)$$

$$\Lambda_\nu^{(r)}=\|\alpha_{k\nu}^{(r)}+\gamma_{k\nu}^{(r)}Q_\nu(|W_\nu|)|W_\nu|^{-1}\|,$$
$$\Xi_\nu^{(r)}=\|\beta_{k\nu}^{(r)}+\delta_{k\nu}^{(r)}L_\nu(|W_\nu'|)|W_\nu'|^{-1},\|,\ H_{l_r-0}=\|\alpha_k^{(r)}\|$$

and the vector

$$X_{l_r+0}=\left\|\begin{matrix}0, & 0 & ,\ldots, & E_2\\ g_{r1}, & g_{r2} & ,\ldots, & g_{rr}\end{matrix}\right\|\times\{h_1,h_2,\ldots,h_r\}+H_{l_r+0} \qquad (2.8)$$

where

$$g_{r\nu}=\left\|\begin{matrix}\alpha_{3\nu}^{(r)}+\gamma_{3\nu}^{(r)}Q_\nu(|W_\nu|)|W_\nu|^{-1}, & \beta_{3\nu}^{(r)}+\delta_{3\nu}^{(r)}L_\nu(|W_\nu'|)|W_\nu'|^{-1}\\ \alpha_{4\nu}^{(r)}+\gamma_{4\nu}^{(r)}Q_\nu(|W_\nu|)|W_\nu|^{-1}, & \beta_{4\nu}^{(r)}+\delta_{4\nu}^{(r)}L_\nu(|W_\nu'|)|W_\nu'|^{-1}\end{matrix}\right\| \quad (\nu=1,\ldots,r-1)$$

$$g_{rr}=\left\|\begin{matrix}0, & \omega^2K_r+i\omega\varkappa_{1r}+L_r(|W_r'|)|W_r'|^{-1}\\ m_r\omega^2-i\omega\varkappa_{2r}-Q_r(|W_r|)|W_r|^{-1}, & 0\end{matrix}\right\|, \qquad h_\nu=\left\|\begin{matrix}W_\nu\\ W_\nu'\end{matrix}\right\|$$

$$H_{l_r+0}=\{0,0,\alpha_3^{(r)},\alpha_4^{(r)}+m_r\varepsilon\omega^2\exp i\theta_r\}$$

The functions $\alpha_{k\nu}^{(r)}$, $\beta_{k\nu}^{(r)}$, $\alpha_k^{(r)}$ of the angular velocity ω in the above formulas can be determined by the familiar methods of the linear theory of the bending oscillations of rotors.

Let us denote the first and second rows of the fourth-order identity matrix E_4 by e_1 and e_2, respectively. From the conditions of equality of the complex deflections and angles of rotation in the cross sections $x=l_r-0$ and $x=l_r+0$ we obtain the system of 2 $(s-1)$ equations

$$e_1(X_{l_r+0}-X_{l_r-0})=e_2(X_{l_r+0}-X_{l_r-0})=0 \qquad (r=2,\ldots,s) \qquad (2.9)$$

Let us set $W_r=A_r\exp i\varphi_r$, $W_r'=B_r\exp i\psi_r$, where the amplitudes A_r, B_r and the phases φ_r, ψ_r are the unknown (required) real numbers; Eqs. (2.9) now become

$$\sum_{\nu=1}^{r-1}\{[A_\nu\alpha_{1\nu}^{(r)}+\gamma_{1\nu}^{(r)}Q_\nu(A_\nu)]e^{i\varphi_\nu}+[B_\nu\beta_{1\nu}^{(r)}+\delta_{1\nu}^{(r)}L_\nu(B_\nu)]e^{i\psi_\nu}\}=A_re^{i\varphi_r}$$

$$\sum_{\nu=1}^{r-1}\{[A_\nu\alpha_{2\nu}^{(r)}+\gamma_{2\nu}^{(r)}Q_\nu(A_\nu)]e^{i\varphi_\nu}+[B_\nu\beta_{2\nu}^{(r)}+\delta_{2\nu}^{(r)}L_\nu(B_\nu)]e^{i\psi_\nu}\}=B_re^{i\psi_r}$$

Let us add to them the two boundary conditions at the right end of the rotor. Separating the real and imaginary parts, we obtain $4s$ equations from which we can determine the amplitudes A_r, B_r and the phases φ_r, ψ_r $(r=1,\ldots,s)$. There can be several numbers of solutions; each of them corresponds to a specific shape $W(x)=A(x)\exp i\varphi(x)$ of the elastic rotor axis and to a specific periodic oscillation

$$u_1^\circ=A(x)\cos[\omega t+\varphi(x)], \qquad u_2^\circ=A(x)\sin[\omega t+\varphi(x)] \qquad (2.10)$$

Exact closed-form solutions (2.10) should be tested for stability. The small perturbations $\xi_j=u_j-u_j^\circ$ $(j=1,2)$ must satisfy

1) the following differential equations in each domain:

$$EI\frac{\partial^4\xi_j}{\partial x^4}+\rho\frac{\partial^2\xi_j}{\partial t^2}+\varkappa\frac{\partial\xi_j}{\partial t}=0 \qquad (2.11)$$

2) the following matching conditions at the nonlinear supports for $x=l_r$ $(r=1,\ldots,s)$:

$$\xi_j(l_r+0)=\xi_j(l_r-0), \qquad \frac{\partial\xi_j(l_r+0)}{\partial x}=\frac{\partial\xi_j(l_r-0)}{\partial x}$$

$$EI\left[\frac{\partial^2\xi_j(l_r+0)}{\partial x^2}-\frac{\partial^2\xi_j(l_r-0)}{\partial x^2}\right]=K_{1r}\frac{\partial^3\xi_j}{\partial x\partial t^2}-i^{2j}\omega K_{0r}\frac{\partial^2\xi_{j\pm1}}{\partial x\partial t}+\varkappa_{1r}\frac{\partial\xi_j}{\partial x\partial t}+$$
$$+\frac{1}{2}\frac{\partial\xi_j}{\partial x}\left[\frac{1}{B_r}L_r(B_r)+\frac{\partial L_r(B_r)}{\partial B_r}\right]+\frac{1}{2}\left[\frac{\partial\xi_j}{\partial x}i^{2j}\cos2(\omega t+\psi_r)-\right.$$
$$\left.-\frac{\partial\xi_{j\pm1}}{\partial x}\sin2(\omega t+\psi_r)\right]\left[\frac{1}{B_r}L_r(B_r)-\frac{\partial L_r(B_r)}{\partial B_r}\right]$$

$$EI\left[\frac{\partial^3\xi_j(l_r+0)}{\partial x^3}-\frac{\partial^3\xi_j(l_r-0)}{\partial x^3}\right]=-m_r\frac{\partial^2\xi_j}{\partial t^2}-\varkappa_{2r}\frac{\partial\xi_j}{\partial t}-\frac{1}{2}\xi_j\left[\frac{1}{A_r}Q_r(A_r)+\frac{\partial Q_r}{\partial A_r}\right]-$$
$$-\frac{1}{2}[\xi_ji^{2j}\cos2(\omega t+\varphi_r)-\xi_{j\pm1}\sin2(\omega t+\varphi_r)]\left[\frac{1}{A_r}Q_r(A_r)-\frac{\partial Q_r(A_r)}{\partial A_r}\right] \quad (j=1,2) \qquad (2.12)$$

where the plus sign in the subscripts applies for $j = 1$ and the minus sign for $j = 2$;

3) the same boundary conditions as the functions $u_j(x, t)$;

4) the same (but homogeneous) linear matching conditions as the functions $u_j(x, t)$ at the sites of the concentrated masses, rigid or elasticomassive supports with linear characteristics, etc.

It is easy to show that relations (2.12) are equations in variations for nonlinear matching conditions (2.2) for $x = l_r$. Unlike the remaining boundary conditions for the perturbations ξ_j, (2.12) contains terms with π / ω-periodic coefficients. If the modulation of these coefficients is not large, then characteristic exponents (1.15) can be determined by the method of Sect. 1 with allowance for the appended Note.

Translated by A.Y.

PROBLEMS OF OPTIMIZATION WITH CONSTRAINTS IMPOSED ON THE PHASE COORDINATES

PMM Vol. 34, №1, 1970, pp. 127-131
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(Received April 1, 1969)

We consider the problems of optimization of control processes with first and higher order constraints imposed on the phase coordinates [1-3]. We establish conditions which make easier the determination of the point at which the phase trajectory leaves the boundary of the region of admissible variation of coordinates.

1. Statement of the problem. The problem studied in [2, 4] was the following. Out of the continuous functions $x_s(t)$, $(s = 1,\ldots, n)$ possessing piece-wise continuous derivatives $x_s^{\cdot}(t)$ and out of the piece-wise continuous controls $u_k(t)$, $(k = 1,\ldots, m)$ satisfying the differential equations

$$g_s = x_s^{\cdot} - f_s(x, u, t) = 0 \qquad (s = 1,\ldots n) \tag{1.1}$$

on the interval $[t_0, T]$, the relations

$$\psi_k = \psi_k(x, u, t) = 0 \quad (k = 1,\ldots, r < m) \tag{1.2}$$

the inequality

$$\vartheta(x) \leqslant 0 \tag{1.3}$$

at the ends of the segment $[t_0, T]$ and the conditions

$$\varphi_l = \varphi_l[x(t_0), t_0, x(T), T] = 0 \ (l = 1,\ldots, p \leqslant 2n + 1) \tag{1.4}$$

to find those, which minimize the functional

$$I = g[x(t_0), t_0, x(T), T] + \int_{t_0}^{T} f_0(x, u, t)\, dt \tag{1.5}$$

Here x and u denote the respective sets of phase coordinates $x_1,\ldots x_n$ and controls $u_1,\ldots u_m$.

In such problems the optimal trajectory may include segments belonging to the boundary of the region defined by the inequality (1.3). In the following, we shall concentrate our attention on such segments.

If a segment of the trajectory lying on the interval $[t_1, t_2]$ belongs to the boundary of